Example The hypotenuse of an isosceles right triangle is increasing at a rate of 1 in/min, and it currently has a length of 6. How fast is the Area increasing?

Given h, $\frac{dh}{dt}$, we want to find $\frac{dA}{dt}$.

$\Rightarrow$ need an equation with A & h in it.

$A = \frac{1}{2}x^2$, Area of triangle

$x^2 + x^2 = h^2$ Pythag. Thm.

$\Rightarrow 2x^2 = h^2$

$\Rightarrow x^2 = \frac{h^2}{2}$

$$\Rightarrow A = \frac{1}{4}h^2$$

$$\Rightarrow \frac{dA}{dt} = \frac{1}{4}2h\frac{dh}{dt} = \frac{2}{4}\cdot 6 \cdot 1 \ \frac{in^2}{min}$$

$$= \boxed{3 \ \frac{in^2}{min}}.$$

Example

Given $\frac{dV}{dt} = -3\frac{ft^3}{min}$

$h = 20ft$

What is $\frac{dh}{dt}$ in $\frac{in}{min}$?

$V = \frac{1}{3}\pi r^2 h$ volume of a cone

Similar triangle

$\frac{r}{h} = \frac{10}{30} = \frac{1}{3} \Rightarrow r = \frac{1}{3}h$

$$\Rightarrow V = \frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 \cdot h = \frac{\pi}{27}h^3.$$

$$\frac{dV}{dt} = \frac{3\pi}{27} h^2 \frac{dh}{dt} = \ldots = -.0214\ldots \frac{ft}{min}$$

$$-.0214 \frac{ft}{min} \cdot \frac{12\ in}{ft} = \boxed{}\ \frac{in}{min}$$

Examples

(1) Let $g(s) = \dfrac{\arctan(s)}{s^2+1}$. Find and classify the critical points of g.

Critical points: values of x where $g'(x) = 0$ or is undefined.

classify (as local max, local min, or saddle pt)

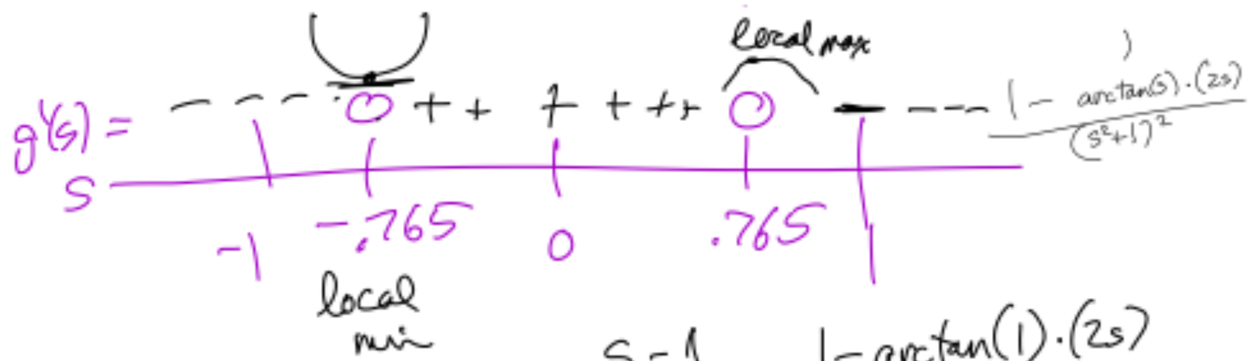
$$g'(s) = \frac{(s^2+1)\frac{1}{1+s^2} - \arctan(s)(2s)}{(s^2+1)^2}$$

$$= \frac{1 - \arctan(s)\cdot(2s)}{(s^2+1)^2} = 0$$

$$\Longleftrightarrow \quad 1 - \arctan(s)(2s) = 0.$$

We have to solve for this numerically.

Sagemath. $\Rightarrow$ $s = -.765$ or $.765$. only critical pts.



$s=1$ $\quad \dfrac{1-\arctan(1)\cdot(2s)}{(s^2+1)^2}$

$$\frac{1-\frac{\pi}{4}\cdot 2}{2^2} = \frac{1-\frac{\pi}{2}}{2^2}$$

3.14

$s=0$ $\quad \dfrac{1-\arctan(0)\cdot 2(0)}{(0^2+1)^2} = 1$

$s=-1$ $\quad \dfrac{1-\arctan(-1)\,(2(-1))}{((-1)^2+1)^2}$

$$= \frac{1-\left(\frac{-\pi}{4}\right)(-2)}{2^2} = \frac{1-\frac{\pi}{2}}{2^2} = -$$

$\therefore$ crit pts are $x=-.765$ (local min)
$x = .765$ (local max).

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② Find where $y=B(x)$ is concave up & down, where $B(x) = 3x^2-45x+x^3+27$.

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cc up $\Leftrightarrow$ $B''(x)$ is positive
cc down $\Leftrightarrow$ $B''(x)$ is negative.

$$B'(x) = 6x - 45 + 3x^2$$

$$B''(x) = 6 + 6x = 0$$

$$6(1+x) = 0 \Rightarrow x = -1.$$

place when $B''(x) = 0$

cc down — cc up

$B''(x) = - - - \; 0 \; + + +$

$x = \quad -2 \quad -1 \quad 0$

$x = -1$ is an inflection pt.

$B(x)$ is cc up for $x \in (-1, \infty)$

$B(x)$ is cc down for $x \in (-\infty, -1)$

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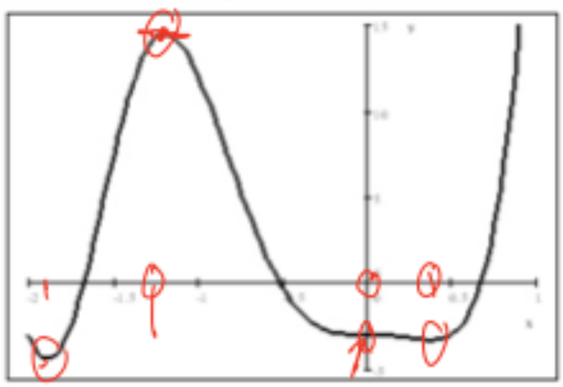
③ Exercise 4.3.

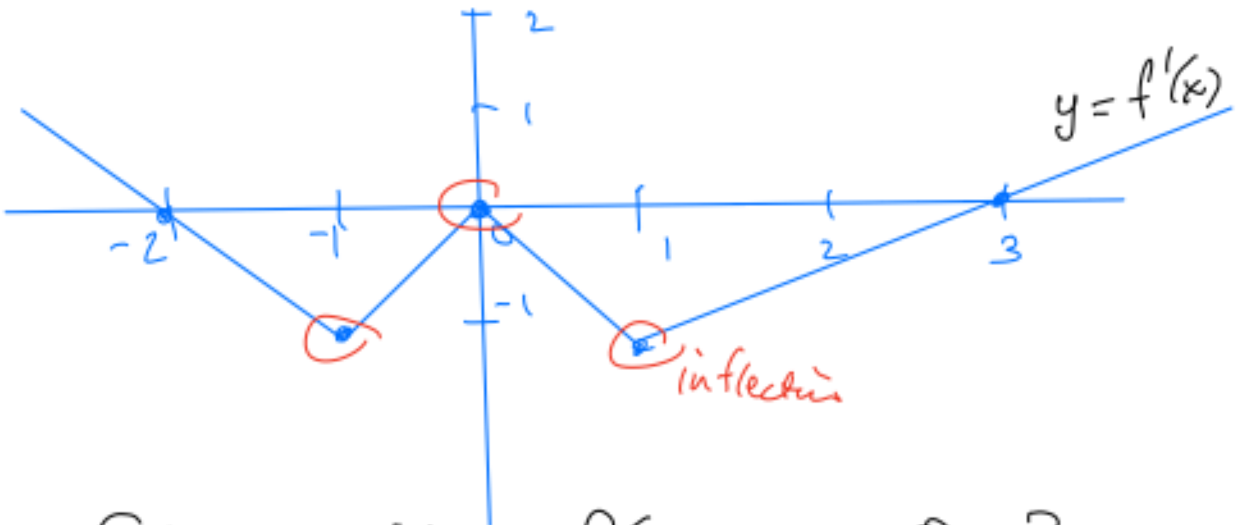
4.3 Indicate all of the critical points on the graph below, and in each case determine the type of critical point.



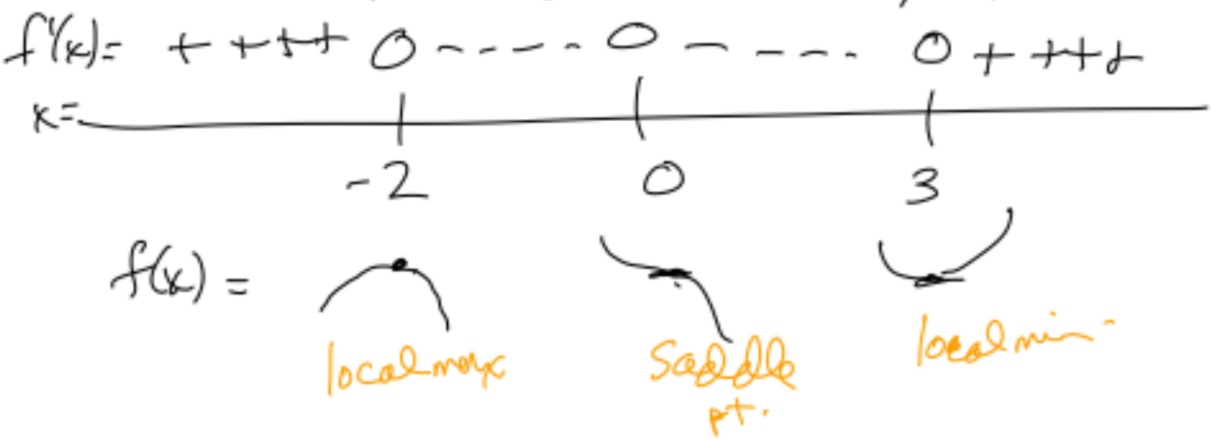
$x = -1.9, -1.25, 0, 0.3$

local min — local max — saddle pt — local min

(4) Below is graphed $y = f'(x)$. Find the critical points & inflection pts, and classify the critical points.



Critical points of $f(x)$: $-2, 0, 3$



inflection pts — concavity changes.

CC up → increasing slope

cc down → decreasing slope

CC up $\Leftrightarrow$ f' increasing $\Leftrightarrow$ $x \in (-1, 0) \cup (1, \infty)$

cc down $\Leftrightarrow$ f' decreasing $\Leftrightarrow$ $x \in (-\infty, -1) \cup (0, 1)$.

inflection pts: $x = -1, 0, 1$

(same as local mins & maxs of $f'(x)$.)

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⑤ Find the absolute max & min of

$$g(s) = 2^{s+3} - \frac{2^{3s-4}}{3} \text{ on the interval } [0,4].$$

Theorem: Extreme Value Theorem. The absolute min & max of a continuous function on a closed interval is always achieved at a point of that interval.

- If $f: [a,b] \to \mathbb{R}$ is a differentiable function, then the absolute max & min can either be:

① a critical point in (a,b)

or

② a or b.

Solution to ⑤: $g'(s) = (\ln 2) 2^{s+3} \cdot (1)$

$$- \frac{1}{\cancel{3}} (\ln 2)(2^{3s-4}) \cdot \cancel{3}$$

$\Rightarrow \quad g'(s) = \ln(2)\, 2^{s+3} - \ln(2)\, 2^{3s-4}$

cr pts: $\quad g'(s) = 0 = \underset{\substack{\neq \\ 0}}{(\ln 2)}\, \underbrace{2^{s}}_{+}\left(2^{3} - 2^{2s-4}\right)$

$\Leftrightarrow \quad 2^{3} - 2^{2s-4} = 0 \Leftrightarrow 2^{3} = 2^{2s-4}$

$\Leftrightarrow \quad 3 = 2s - 4 \Rightarrow 7 = 2s \Rightarrow s = \frac{7}{2}$

only 1 cpt.

Possible abs max & min are

$s = 0, \frac{7}{2}, 4$

s	$g(s) = 2^{s+3} - \frac{2^{3s-4}}{3}$
0	$2^{3} - \frac{2^{-4}}{3} = 8 - \frac{\frac{1}{16}}{3} = 8 - \frac{1}{48} = 7\frac{47}{48}$
$\frac{7}{2}$	$2^{7/2+3} - \frac{2^{3(\frac{7}{2})-4}}{3} = 2^{6.5} - \frac{2^{6.5}}{3} = \frac{2}{3}2^{6.5} = \frac{2^{7.5}}{3}$ 60.32
4	$2^{7} - \frac{2^{12-4}}{3} = 2^{7} - \frac{2^{8}}{3} =$ ~~negative~~ 42.67

critical values

Abs max $x = \frac{7}{2}$ (60.32)

Abs min $x = 0$ $\left(7\frac{47}{48}\right)$

matches desmos graph!